The study of chaotic attractors of nonlinear systems by path integration

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ABSTRACT: The paper discusses the possibility of using the methods of stochastic differential equations to study the dynamic response of nonlinear oscillators driven by harmonic or perturbed harmonic excitations. The reported work continues and extends previous work done by the authors using a numerical path integration scheme for solving stochastic differential equations. It is shown that this solution method may offer an attractive means of making response predictions for nonlinear dynamic systems of the kind considered.

1 INTRODUCTION

In this paper the primary goal is to continue the investigation of the capability of the numerical path integration (PI) method to provide approximations of the strange or chaotic attractors of nonlinear oscillators driven by harmonic and perturbed harmonic excitations.

The PI method can be used to solve stochastic differential equations. However, this requires the presence of Gaussian white noise as an additional excitation source. Hence, in order to be able to use the PI method, the dynamic system is modified by introducing a stochastic perturbation in the form of additive Gaussian white noise. The state space vector of the latter system can then often be represented as a Markov diffusion process. The joint probability density function (PDF) of the state space vector of this Markov process, which is provided by the PI method, is closely related to the corresponding chaotic attractor of the underlying deterministic system, which is obtained as a limit of the joint PDF as the intensity of the added noise goes to zero. Previous studies have indicated that the obtained PDF for small added noise can be effectively used for prediction purposes (Naess 2000). In the proposed paper, such possibilities will be further explored and demonstrated.

2 THE NONLINEAR OSCILLATOR

The nonlinear dynamic systems to be studied numerically in this work belong to the general class of Duffing oscillators with an equation of motion that can be expressed as follows

\[ \ddot{X}_t + c \dot{X}_t + k_1 X_t + k_3 X_t^3 = a \cos(\omega t) + b N_t \quad (1) \]

where \( a, b, c \) are non-negative constants, and \( k_1, k_3 \) are suitable constants. The stochastic process \( N_t \) is a standard Gaussian white noise, that is, \( E[N_t N_{t+\tau}] = \delta(\tau) \), where \( \delta(\cdot) \) denotes Dirac’s delta function. \( N_t \) can be considered as the formal derivative of a standard Brownian motion (Wong and Hajek 1985).

Among the nonlinear oscillators with harmonic excitation \((a > 0, \ b = 0)\) exhibiting chaotic
behaviour, the Duffing class plays a central role (Holmes 1979). The following specific oscillator is subjected to a further study here

\[
\ddot{X}_t + \frac{1}{25} \dot{X}_t - \frac{1}{5} X_t + \frac{8}{15} X_t^3 = a \cos(\omega t) + b N_t
\]  

(2)

Note that since \( k_1 < 0 \) and \( k_3 > 0 \), the oscillator becomes bistable, that is, it has two equilibrium points. This oscillator has been extensively studied by Seydel (1985) for the case when \( a = 2/5 \) and \( b = 0 \), and he shows that the oscillator has an extremely rich bifurcation behaviour in the frequency range \( 0 < \omega < 0.5 \). In the present paper we shall investigate the situation with \( b > 0 \) to see to what extent the addition of white noise will alter the character of the response.

3 THE STOCHASTIC DIFFERENTIAL EQUATION AND ITS SOLUTION

For \( b > 0 \), equations (1) and (2) can be written as a stochastic differential equation (SDE), which assumes the form

\[
dY_t = \alpha(Y_t, t) \, dt + \beta \, dB_t
\]  

(3)

where \( Y_t = (X_t, \dot{X}_t)^T \) is the 2D state space vector process, and \( B_t \) is a standard (unit) scalar Brownian motion process (Wong and Hajek 1985). It is seen that \( \alpha(\cdot) = (\alpha_1(\cdot), \alpha_2(\cdot))^T \), where \( \alpha_1(Y_t, t) = -c \dot{X}_t - k_1 X_t - k_3 X_t^3 + a \cos(\omega t) \), \( \beta = (0, b)^T \).

As discussed by Naess and Moe (2000), an efficient numerical solution of the SDE (3) can be obtained by adopting a Runge-Kutta-Maruyama (RKM) discretization approximation

\[
Y_t = Y_{t_0} + \sum_{k=1}^{n} \left( \frac{r(Y_{t_{k-1}}, t_k^t) \Delta t}{r(Y_{t_{k-1}}, t_k^t)} + \beta \Delta W_t \right)
\]  

(4)

Since the Wiener process has independent increments, it follows from equation (4) that the sequence \( \{Y_{n\Delta t}\}_{n=0}^{\infty} \) is a Markov chain. For sufficiently small \( \Delta t \), this Markov chain will approximate the continuous time Markov process solution of the SDE (3). It is also observed from equation (4) that \( p(y, t | y', t') \), which denotes the conditional PDF of \( Y_t \) given that \( Y_t = y', t' < t \), is a Gaussian PDF since \( \Delta W_t \) is a Gaussian variable for every \( t' \). For small time increments \( \Delta t = t - t' \), \( p(y, t | y', t') \) will be referred to as the incremental transition probability density function (TPD).

From the assumptions above we may now write the TPD corresponding to equation (4)

\[
p(y, t | y', t') = \delta(y - y' - r_1(y') \Delta t) \quad \text{and} \quad \tilde{p}(y_2 | y', t') = \frac{1}{\sqrt{2\pi b^2 \Delta t}} \cdot \exp \left\{ -\frac{(y_2 - y' - r_2(y') \Delta t)^2}{2 b^2 \Delta t} \right\}
\]  

(5)

Hence, \( p(y, t | y', t') \) is a degenerate two-dimensional Gaussian PDF.

The path integration principle is based on the following basic relation

\[
p(y, t) = \int_{\Omega} p(y, t | y', t') p(y', t') \, dy'
\]  

(7)

where \( \Omega = \text{state space} \). For a numerical solution of an SDE, it will be shown that the TPD can always be given as an analytical, closed form expression if the time increment is sufficiently small. Hence, if an initial PDF, \( p_0(y) = p(y, t_0) \), say, is given, then equation (7) can be invoked repeatedly to produce the time evolution after \( t_0 \) of \( p(y, t) \). The numerical implementation of the PI procedure we have developed for problems of state space dimension three or less has been described in detail by Naess and Johnsen (1993), Naess and Moe (2000). It is based upon using a splines interpolation scheme for continuous representation of the joint PDF \( p(y, t) \) after each iteration of equation (7), which provides the values of the PDF at a discrete set of grid points. Experience indicates that the success of the numerical PI relies heavily on the accuracy of the continuous representation.

4 RESPONSE PREDICTIONS

Typical response predictions in engineering analyses of dynamic structures involve estimation of the crossing rate statistics of prescribed response levels. We shall demonstrate how this can be achieved for nonlinear oscillators described by equation (1). As shown above, there is a joint PDF for the state space vector \( Y_t = (X_t, \dot{X}_t)^T \), which is denoted here by \( f_{X\dot{X}}(x, \dot{x}; t) \). It is assumed that the mean up-crossing rate of a level \( \xi \) by \( X_t \) at time \( t \) can be calculated by using the Rice formula
\( \nu^+_X(\xi; t) = \int_0^\infty f_{X\bar{X}}(\xi, s; t) \, s \, ds \) \tag{8}

As discussed in Naess (2000), in the case of positive damping, that is \( c > 0 \), it can be assumed that \( f_{X\bar{X}}(x, \bar{x}; t) = f_{X\bar{X}}(x, \bar{x}; t + t_p) \), where \( t_p = 2\pi/\omega \), when \( t \geq t' \) for a suitably large \( t' \). That is, the joint density \( f_{X\bar{X}}(x, \bar{x}; t) \) eventually becomes periodic after transient effects have died down.

It follows that the mean number of upcrossings of the level \( \xi \) over one period \( t_p \) by \( X_t \) for \( t > t' \), which we denote by \( n^+_p(\xi) \) is given as

\[
  n^+_p(\xi) = \int_{t'}^{t'+t_p} \nu^+_X(\xi; t) \, dt = t_p \int_0^\infty f_{X\bar{X}}(\xi, s) \, s \, ds
\]

where the averaged density

\[
  \bar{f}_{X\bar{X}}(x, \bar{x}) = \frac{1}{t_p} \int_{t'}^{t'+t_p} f_{X\bar{X}}(x, \bar{x}; t) \, dt
\]

has been introduced.

Hence it is seen that for practical estimation of crossing rate statistics, it is often the average density that is important and not the density at a specific point in time. This observation has practical consequences. While the instantaneous joint PDF for an underlying chaotic system perturbed by small Gaussian white noise reflects the complexity of the chaotic attractor produced by the Poincaré map, see Figures (11) and (12) the corresponding averaged density generally has a simpler structure. And, significantly, the concept of stationary density makes perfect sense after averaging.

5 NUMERICAL RESULTS
To explore the effect of additive white noise to the Duffing equation (2), that is, \( b > 0 \), we have calculated the stationary average joint PDF of the Markov chain associated with equation (4) as given by equation (2) for \( a = 0.4 \) and \( b = 0.005 \) for four values of the frequency of the harmonic part of the excitation. The choice of \( b \)-value is based on two considerations. It should be much smaller than \( a \), but not so small as to make numerical calculations of the joint PDF of the state space vector too demanding. This is particularly relevant for the cases when the deterministic oscillator has periodic response leading to a very singular behaviour of the joint PDF for small values of \( b \).

5.1 Example 1 - \( \omega = 0.04 \)
For the case \( \omega = 0.04 \), Seydel (1985) shows that the deterministic oscillator has a chaotic or strange attractor, but with a very simple structure similar to the cross section of a torus. Why this is so can be better understood by looking at the time history of the oscillator response, cf. Figure (1). In Figure (2), the corresponding time history of the response with added white noise has been plotted. The plots in Figures (1) and (2) are difficult to distinguish qualitatively. Both figures convey the impression of a very regular response consisting of two periodic components, one rapid and one slow. The characteristics of the response are also clearly displayed by the joint PDF plotted in Figure (3). The rapid, small amplitude oscillations manifest themselves by the presence of the two conspicuous craters symmetrically located wrt the origin. The presence of the slow, large amplitude oscillations are evidenced by the non-negligible probability mass connecting the two crater regions.

Note that on all plots, \( x \) denotes displacement, \( y \) denotes velocity.

![Figure 1: Part of the time history of the deterministic response for \( \omega = 0.04 \).](image)

5.2 Example 2 - \( \omega = 0.20 \)
At this driving frequency, the oscillator has a much more pronounced chaotic attractor, which is demonstrated by the Poincaré section plot of Figure (4). Figure (5) shows a contour plot of the
corresponding instantaneous pdf of the stochastic oscillator at the end of 30 periods of the harmonic excitation, and it is seen that the structure of the chaotic attractor is reflected in the pdf.

Examples of response time histories of the deterministic and stochastic oscillator are presented in Figures (6) and (7). While there appears to be no qualitative difference between the two time histories, they both convey the impression of irregularity, in agreement with the existence of a chaotic attractor for the deterministic oscillator.

Figure (8) shows the corresponding stationary averaged joint PDF. It is clearly seen that there is no conspicuous periodic behaviour of this oscillator. Before we comment in more detail on this case, it is expedient to first have a look at the next two examples.
Figure 7: Part of the time history of the stochastic response for $\omega = 0.20$.

Figure 9: Part of the time history of the deterministic response for $\omega = 0.28$.

Figure 8: The averaged pdf of the stochastic oscillator for $\omega = 0.20$.

Figure 10: Phase space plot of the deterministic periodic attractor for $\omega = 0.28$.

5.3 Example 3 - $\omega = 0.28$

For this driving frequency, the deterministic oscillator has a periodic stationary response, which is easily recognized from the time history shown in Figure (9), and the phase space plot of the stationary response shown in Figure (10). The corresponding stationary averaged joint PDF of the stochastic oscillator is plotted in Figure (11), which clearly shows the periodic behaviour of the oscillator.

5.4 Example 4 - $\omega = 0.32$

In this case, the deterministic oscillator again displays chaotic behaviour. An example of a Poincaré section plot for this case is shown in Figure (12). Figure (13) shows a contour plot of the corresponding instantaneous PDF of the state space vector of the stochastic oscillator at the end of 30 periods of the harmonic forcing function. It is seen that the structure of the chaotic attractor is reflected in the PDF. Also in this case there is apparently no qualitative difference between the determinis-
tic and stochastic response time histories, see Figures (14) and (15). The corresponding stationary average joint PDF is plotted in Figure (16).

Figure 12: A Poincare section plot of the deterministic oscillator for \( \omega = 0.32 \).

Figure 13: A contour plot of the joint PDF of the state space vector of the stochastic oscillator at the end of 30 periods of the harmonic forcing function for \( \omega = 0.32 \).

6 CONCLUDING REMARKS
In this paper we have pursued a study of the chaotic response of a harmonically forced Duffing type oscillator. Supporting previous findings (Naess 2000), we have shown that important aspects of the response of engineering interest appear to be largely unchanged by additive white noise. This opens for an alternative avenue of investigation of the response properties of nonlinear chaotic systems by exploiting the numerical path integration method. In many respects, this approach appears to be a more effective means of analysis than Monte Carlo simulation methods.

Note that the large displacement responses of
the oscillator is not much influenced by whether the oscillator is in the chaotic or periodic regime. This is plausible from a recognition that the chaotic response of the bistable Duffing oscillator is caused by an unpredictable transition between the two potential wells of the oscillator.

REFERENCES